JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **114**, 17–36 (1986)

Finite-Dimensional Control of Distributed Parameter Systems by Galerkin Approximation of Infinite Dimensional Controllers*

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Submitted by G. Leitmann

The Galerkin method is presented as a way to develop finite-dimensional controllers for linear distributed parameter systems (DPS). The *direct* approach approximates the open-loop DPS and then generates the controller from this approximation; the *indirect* approach approximates the infinite-dimensional stabilizing controller. Conditions under which they produce equivalent controllers in the sense of closed-loop stability are presented. The indirect approach is shown to converge to the stable closed-loop system consisting of DPS and infinite-dimensional controller; conditions are presented on the behavior of the Galerkin method for the open-loop DPS which guarantee closed-loop stability for large enough finite-dimensional approximations. © 1986 Academic Press Inc.

1.0. INTRODUCTION

Taking into account the distributed nature of the dynamics in many engineering systems, e.g., large space structures, chemical processes, and fusion plasma reactors, we must model them with partial differential equations. Such DPS require an infinite-dimensional state space for a proper description of their dynamical behavior. However, feedback control of DPS requires a finite-dimensional algorithm which can process information with an on-line digital computer from a few (P) sensors to produce control commands for a small number (M) of actuators.

* This research was supported in part by the National Science Foundation under Grant ECS-80-16173, the National Aeronautics and Space Administration under Grant NAG-1-171, and the Air Force Office of Scientific Research under Grant AFOSR-83-0124. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the author and do not necessarily reflect the views of NSF, NASA, or AFOSR.

0022-247X/86 $3.00

In previous work, e.g., [1]–[3], we have focused on what can be accomplished by finite-dimensional control of infinite-dimensional systems both in stability and performance. We have stressed the need and developed bounds for closed-loop stability analysis when reduced-order controllers are used with DPS. Also, we have characterized the finite-dimensional exponentially stabilizing controllers for linear DPS in the time-domain via operator semigroup theory; this work deals with discrete and continuous-time controllers of fixed, finite order [2, 4].

In this paper, we present two basic approaches (direct and indirect) for model reduction of the DPS via the Galerkin method. We show conditions under which they are equivalent in the sense that they both produce finite-dimensional controllers which stabilize the infinite-dimensional DPS. In [5], we answered the question: when will a finite-dimensional controller based on a direct Galerkin model reduction of a linear DPS produce stable closed-loop control? This is related to the results of [6] where it is shown that a sufficiently large-dimensional approximation of a linear DPS will always be adequate for stable controller design. Here we approach the problem from a different direction. There is almost always an infinite-dimensional stabilizing controller for the DPS; consequently, we investigate what happens when the indirect Galerkin approximation, i.e., approximation of this controller, is used to obtain a finite-dimensional controller. This follows the design philosophy: *keep it infinite as long as possible*. We present conditions under which this approach leads to a finite-dimensional stabilizing controller for the DPS.

In Section 2.0, we present DPS preliminaries. In Section 3.0, we examine the direct and indirect Galerkin approaches and show conditions for their equivalence. We present our main results on the approximation of infinite-dimensional controllers for DPS in Section 4.0.

2.0. DPS Preliminaries

The linear DPS of interest will be modeled by the following *state space form*:

$$\begin{aligned} \frac{\partial v(t)}{\partial t} &= Av(t) + Bf(t); \qquad v(0) = v_0, \\ y(t) &= Cv(t), \end{aligned} \tag{2.1}$$

where the *state* $v(t)$ is in an infinite-dimensional Hilbert space H with inner product $(\cdot, \cdot)$ and corresponding norm $\|\cdot\|$. The input–output operators B and C have finite ranks M and P, respectively, and $f(t)$, $y(t)$ represent the inputs for M linear actuators and the outputs from P linear sensors, respectively. Thus,

$$Bf(t) = \sum_{i=1}^{M} b_i f_i(t) \tag{2.2}$$

and

$$y(t) = [y(t), \ldots, y_P(t)]^{\mathrm{T}} \quad \text{with} \quad y_j(t) = (c_j, v(t)); \quad 1 \leqslant j \leqslant P, \tag{2.3}$$

where b_i and c_j belong to H. In finite-dimensional theory, A would be a matrix, but here the operator A is a closed, linear, unbounded differential operator with domain $D(A)$ dense in H. Furthermore, (2.1)–(2.3) represent some *well-posed* physical system, which in mathematical terms is the *weak formulation* of (2.1):

$$\begin{aligned} v(t) &= U(t)\, v_0 + \int_0^t U(t-\tau)\, Bf(\tau)\, d\tau, \\ y(t) &= Cv(t);\ t \geqslant 0, \end{aligned} \tag{2.4}$$

where v_0 is *any* initial state in H and $U(t)$ is the C_0-semigroup of bounded operators generated on H by A. This latter means

$$U(t+\tau) = U(t)\, U(\tau); \qquad t \geqslant 0, \quad \tau \geqslant 0 \tag{2.5a}$$

$$U(0) = I, \tag{2.5b}$$

$$\lim_{t \to 0^+} [U(t) - I]\, v = 0; \qquad v \text{ in } H, \tag{2.5c}$$

$$Av = \lim_{t \to 0^+} \left[\frac{U(t) - I}{t} \right] v; \qquad v \text{ in } D(A). \tag{2.5d}$$

Note that the semigroup $U(t)$ evolves the initial conditions v_0 forward in time. When v_0 is in $D(A)$ and $f(t)$ has continuous first derivative, $v(t)$ also is differentiable, lies in $D(A)$ for $t \geqslant 0$, and satisfies (2.1). However, any v_0 in H and any square-integrable $f(t)$ will satisfy the weak formulation (2.4) and yield states $v(t)$ in H for all $t \geqslant 0$. Consequently, (2.4) is much easier to work with in infinite dimensions and is more likely to represent the actual physical system being modeled by (2.1).

This form, (2.1) or (2.4), models most practical *interior control problems* for linear DPS where the actuator and sensor influence functions are given by b_i and c_j, respectively. *Linear boundary control problems* for DPS have a somewhat different form from (2.1); however, they can usually be converted to equivalent interior control problems which do look like (2.1) [2]. Therefore, we will focus on the form (2.1) without any loss of generality for linear DPS problems.

The *Hille–Yosida theorem* provides conditions under which a closed opeator A generates a C_0-semigroup $U(t)$ satisfying

$$\|U(t)\| \leqslant Ke^{-\sigma t}, \qquad t \geqslant 0, \tag{2.6}$$

where $K \geqslant 1$ and σ real. The necessary and sufficient conditions are given for the resolvent operator $R(\lambda, A) \equiv (\lambda I - A)^{-1}$,

$$\|R(\lambda, A)^n\| \leqslant \frac{K}{(\lambda + \sigma)^n}; \qquad n = 1, 2, ..., \tag{2.7}$$

for all real $\lambda > -\sigma$ in the resolvent set of A, $\rho(A) = \{\lambda \text{ complex} \mid R(\lambda, A) \text{ is a bounded operator on } H\}$. The spectrum of A, $\sigma(A) = \rho(A)^c$ is much more complicated in infinite dimensions, but, in finite dimensions, it consists only of the (finite number of) eigenvalues of A. We say that A *is exponentially stable* when $\sigma > 0$ in (2.6), i.e., the semigroup $U(t)$ generated by A decays exponentially at the rate σ. There are many other types of stability in infinite dimensions, but no others provide the safety of a *stability margin* σ; therefore, this seems like the kind of stability of most practical interest for engineering applications where there is always some uncertainty in the model of the DPS.

We say that the pair (A, B) in (2.1) is (exponentially) *stabilizable* if there is a bounded linear gain operator $G: H \to R^M$ such that $A + BG$ generates an exponentialy stable C_0-semigroup, i.e., the semigroup satisfies (2.6) with $\sigma > 0$. Similarly, the pair (A, C) in (2.1) is *detectable* if (A^*, C^*) is stabilizable where A^* is the adjoint operator associated with A.

We say that (A, B) in (2.1) has a pair of *stabilizing subspaces* (H_N, H_R) if the following hold:

$$H = H_N \oplus H_R, \tag{2.8a}$$

$$\dim H_N = N < \infty, \tag{2.8b}$$

and $A_0 \equiv A + BG$ generates an exponentially stable C_0-semigroup $U_0(t)$, i.e.,

$$\|U_0(t)\| \leqslant K_0 e^{-\sigma_0 t}; \qquad t \geqslant 0 \tag{2.8c}$$

with $K_0 \geqslant 1$ and $\sigma_0 > 0$, where

$$G = GP_N \qquad (\text{or } GP_R = 0) \tag{2.8d}$$

with (P_N, P_R) the projections defined by (2.8a). Thus, stabilizing subspaces guarantee that the *projection feedback law*

$$f(t) = GP_N v(t) \tag{2.9}$$

can produce an exponentially stable closed-loop system (2.1) and (2.9). Usually, we assume that σ_0 is specified; hence, (2.1) may have stabilizing subspaces for some values σ_0 but not for others (clearly, if it has them for some $\sigma_0 > 0$ then it will have them for all smaller values $0 < \sigma \leqslant \sigma_0$). Of course, it should be noted that (2.9) is an *ideal* control law which cannot in general be generated from the sensor outputs (2.3). Our main result in [2] shows that *every finite-dimensional stabilizing controller must asymptotically reproduce* (2.9) *for a special pair of stabilizing subspaces.*

Next we present some results for infinite-dimensional DPS controllers which are analogous to the finite-dimensional state-space controllers for lumped parameter systems. Unlike their finite-dimensional counterparts, these controllers cannot be implemented with practical computers and devices in general. Nevertheless, such results give further insight into the DPS control problem and are needed in later sections.

The first result gives conditions under which the full state $v(t)$ of the DPS can be recovered asymptotically from the finite number of available measurements $y(t)$ by an *infinite-dimensional state estimator* (Kalman filter of Luenberger observer):

THEOREM 2.1. *If (A, C) is detectable then there is a bounded operator K mapping R^P into $D(A)$ such that the estimated state $\hat{v}(t)$ generated by the* state estimator

$$\begin{aligned} \frac{\partial \hat{v}(t)}{\partial t} &= A\hat{v}(t) + Bf(t) + K(y(t) - \hat{y}(t)), \qquad \hat{v}(0) = 0. \\ \hat{y}(t) &= C\hat{v}(t), \end{aligned} \tag{2.10}$$

converges in norm to the actual state $v(t)$ at an exponential rate (determined by K).

The second result gives conditions under which stability of the DPS may be achieved using the state-estimator (2.10):

THEOREM 2.2. *In addition to the hypothesis of Theorem* 2.1, *if (A, B) is also stabilizable, then there is a bounded operator G from $D(A)$ into R^M such that the control law*:

$$f(t) = G\hat{v}(t), \tag{2.11}$$

where $\hat{v}(t)$ is generated by (2.10), *produces an exponentially stable closed-loop system consisting of* (2.1) *and* (2.10), (2.11).

The proofs for these results are given in [1]; except for some infinite-dimensional technicalities they are the same as those for the finite-dimensional case. Note that for finite-dimensional systems (A, B, C) controllable and observable would be sufficient to satisfy the hypothesis of

Theorems 2.1, 2.2; however, in infinite dimensions this is not the case when controllability and observability are taken in the approximate (and most reasonable) sense of [7, Chap. 4].

Therefore, under the above stabilizability conditions on (A, B, C), a stabilizing controller exists, i.e., (2.10), (2.11); however, this infinite-dimensional controller cannot be implemented. In this paper we shall be concerned with *continuous-time, finite-dimensional, linear controllers* for (2.1) of the form

$$f(t) = L_{11}\, y(t) + L_{12} z(t), \tag{2.12a}$$

$$\dot{z}(t) = L_{21}\, y(t) + L_{22} z(t), \tag{2.12b}$$

where $\dim z = \alpha < \infty$. It is *not* an essential restriction that (2.12) be continuous-time; this is only done for convenience. In the next sections, we will describe methods to generate such controllers for DPS and show conditions for which they will produce a stable closed-loop system.

3.0. Galerkin Approximation of DPS: Direct and Indirect Model Reduction

In general, a *reduced-order model* (ROM) of (2.1) is produced by projecting onto a finite dimensional subspace. Suppose

$$H = H_N \oplus H_R, \tag{3.1}$$

where $H_N \subseteq D(A)$ and $\dim H_N = N < \infty$. Let $v_N = P_N v$ and $v_R = P_R v$, where P_N, P_R are the projections (not necessarily orthogonal) onto H_N, H_R respectively. Then (2.1) decomposes into the following form:

$$\frac{\partial v_N}{\partial t} = A_N v_N + A_{NR} v_R + B_N f, \tag{3.2a}$$

$$\frac{\partial v_R}{\partial t} = A_{RN} v_N + A_R v_R + B_R f, \tag{3.2b}$$

$$y = C_N v_N + C_R v_R, \tag{3.2c}$$

where $v = v_N + v_R$, $A_N = P_N A P_N$, $B_N = P_N B$, $C_N = C P_N$, $A_{NR} = P_N A P_R$, etc. All parameters except A_R, are bounded operators since P_N has finite rank. The ROM is produced by ignoring the residuals v_R in (3.2),

$$\begin{aligned} \frac{\partial v_N}{\partial t} &= A_N v_N + B_N f, \\ y &= C_N v_N. \end{aligned} \tag{3.3}$$

This is a finite-dimensional approximation of (2.1) and the parameters

(A_N, B_N, C_N) may be identified with their corresponding matrices in any appropriate basis of H_N.

In the special case [8, Theorem 6.17, p. 178], where the spectrum of A may be separated into two parts $\sigma(A_N)$ and $\sigma(A_R)$, where $\sigma(A_N)$ consists of N isolated eigenvalues of A which can be separated from the rest of the spectrum $\sigma(A_R)$ by a smooth closed curve in the complex plane, there exist *reducing subspaces* H_N and H_R such that A_N has the spectrum $\sigma(A_N)$ and A_R has the spectrum $\sigma(A_R)$ and these subspaces are A-invariant,

$$A_{NR} = 0 \qquad \text{and} \qquad A_{RN} = 0. \tag{3.4}$$

These are also called *modal subspaces* since $H_N = \mathrm{sp}\{\phi_1, ..., \phi_N\}$, where ϕ_K are the mode shapes or eigenfunctions of the operator A which correspond to the eigenvalues $\lambda_1, ..., \lambda_N$ in $\sigma(A_N)$.

Now we develop two basic procedures for synthesizing *finite-dimensional controllers* for the DPS (2.1):

(1) *direct model reduction*, i.e., perform a model reduction on the DPS (2.1) and synthesize the controller directly from this ROM;

(2) *indirect model reduction*, i.e., perform a model reduction on the infinite-dimensional controller (2.10), (2.11) to obtain a finite-dimensional approximation.

We will use the Galerkin method for model reduction in both cases.

The direct procedure is quite straightforward and is the most natural one to use from a practical standpoint. It requires nothing but ROM information for the controller synthesis and can be carried out even though the conditions for existence of an infinite-dimensional controller are not verified. The indirect procedure requires the existence of an infinite-dimensional controller and some knowledge of the gain operators G and K. When this knowledge is available, it seems reasonable to take advantage of it; the finite-dimensional approximation of the infinite-dimensional controller may perform better.

Clearly, there are technical drawbacks to the indirect procedure while the direct procedure can always be performed. The advantages of the indirect procedure will only be apparent in the analysis of the closed-loop system with such finite-dimensional controllers. At the end of this section, we shall present conditions under which the two procedures yield equivalent controllers.

3.1. *The Galerkin Approximation*

Let H_N be an increasing sequence of finite-dimensional subspaces of the state space H for (2.1),

$$H_N \subseteq H_{N+1} \subseteq \cdots \subseteq H. \tag{3.5}$$

Each subspace H_N has dimension N. We *assume* that each H_N is a subspace of $D(A)$ so that its elements satisfy the boundary conditions for A; however, so called nonconforming elements may be used in the more general case. In the *finite element method* (FEM) each subspace H_N consists of splines (i.e., piecewise-polynomial functions) of fixed degree defined over a mesh (usually, of triangles) laid out to approximately cover the spatial domain Ω of the problem (see [9, Chap. 6]). No matter how irregular the shape of the boundary of Ω such meshes can be fitted very closely; this is one of the principal assets of the FEM. To each mesh, a *normalized mesh parameter* h (where $0 < h \leqslant 1$) is assigned so that the mesh is refined as $h \to 0$ and the dimension N^- of the subspaces increases indefinitely.

Henceforth, we will say a sequence $\{A_N\}_{N=1}$ of linear operators A_N: $H \to H$ *converges strongly* to A, i.e., $A_N \xrightarrow{s} A$, when

$$\lim_{N \to \infty} \|A_N v - Av\| = 0$$

for all v in H.

Let P_N be the *orthogonal* projection from H into H_N; this is called the *Galerkin projection*. The corresponding orthogonal projection onto H_R is called P_R (i.e., $P_R = I - P_N$). The "rate of convergence" of H_N to H is said to be of order q when

$$\|P_R v\| \leqslant Kh^q \tag{3.6}$$

for v in $D(A)$; this rate is related to the ability of splines in H_N to interpolate functions in H. We shall not be concerned with the rate of convergence q; consequently we write (3.6) as

$$\lim_{N \to \infty} \|P_R v\| = 0 \qquad \text{for} \quad v \text{ in } D(A) \tag{3.7}$$

(i.e., $P_N \xrightarrow{s} I$ or $P_R \xrightarrow{s} 0$ in $D(A)$) and suppress the dependence on h for our discussion.

Let $\psi_1(x),..., \psi_N(x)$ form a basis in H_N (i.e., they are linearly independent). These functions are called *patch functions* or *assumed mode shapes*. An approximation of the solution $v(x, t)$ of (2.1) can be formed in H_N by

$$v_N(x, t) = \sum_{k=1}^{N} v_k(t)\, \psi_k(x), \tag{3.8}$$

i.e., assume separation of time and space variables with all spatial variance lumped into the patch functions $\psi_k(x)$. The choice of the coefficients $v_k(t)$ remains; these are obtained by substitution of (3.8) into (2.1):

$$\frac{\partial v_N}{\partial t} = Av_N + Bf + E_N, \tag{3.9}$$

where E_N is the equation error, and the $v_k(t)$'s are chosen so that

$$P_N(E_N) = 0. \tag{3.10}$$

This is called the *Galerkin approximation*; when it is carried out with the subspaces H_N described above, it produces (3.8) where the coefficients $v_k(t)$ are given by the entries of the solution vector $\mathbf{v}_N(t) = [v_1(t),..., v_N(t)]^T$ for the following system of ordinary differential equations:

$$\tilde{M}_N \dot{\mathbf{v}}_N = \tilde{A}_N \mathbf{v}_N + \tilde{B}_N f, \tag{3.11}$$

where $\tilde{M}_N = [(\psi_l, \psi_k)]$, $\tilde{A}_N = [(\psi_l, A\psi_k)]$, and $\tilde{B}_N f = [\psi_l, Bf)]$. The matrix $\tilde{M}_N$ is symmetric and positive definite because $\{\psi_k(x)\}_{k=1}^N$ is linearly independent.

Therefore, (3.11) can be solved uniquely for $\mathbf{y}_N(t)$ whenever $\mathbf{v}_N(0)$ is specified, and hence the Galerkin approximation (3.8) is obtained. It is assumed that $\mathbf{v}_N(0)$ is given by the vector of coefficients of

$$v_N(0) = P_N v_0 \tag{3.12}$$

expanded in the basis $\{\psi_k(x)\}_{k=1}^N$. Note that $v_N \neq P_N v$; however,

$$v_N = P_N v_N. \tag{3.13}$$

The approximation (3.8) is called a *semidiscretization* of (2.1) because time t remains continuous.

It should be noted that to obtain the most analytical benefit from the Galerkin method, the approximation (3.8) should be obtained from the "weak" form of (2.1); however, we omit discussion of this technicality and refer to [9] for further details.

3.2. *Feedback Controllers: Direct Model Reduction*

The *Galerkin reduced-order model* associated with (2.1) is defined on H_N and given by

$$\begin{aligned} \frac{\partial v_N}{\partial t} &= A_N v_N + B_N f, \qquad v_N(0) = P_N v_0, \\ y &= C_N v_N, \end{aligned} \tag{3.14}$$

where (A_N, B_N, C_N) are defined from (3.9), (3.10) using (3.13) to be $A_N \equiv P_N A P_N$, $B_N \equiv P_N B$, and $C_N \equiv C P_N$. Since H_N is a finite-dimensional subspace, (A_N, B_N, C_N) may be identified with their matrices in an appropriate basis of H_N, and (3.14) is equivalent to a lumped parameter, state variable system for which a well developed feedback control theory exists. The controllability and observability of (A_N, B_N, C_N) are easily checked. Henceforth, *for the direct* method (A_N, B_N, C_N) *will be assumed to be stabilizable and detectable for each N.*

The *Galerkin Feedback Controller* is based on the ROM (4.1) and defined by

$$f = \tilde{G}_N \hat{v}_N, \tag{3.15a}$$

$$\frac{\partial \hat{v}_N}{\partial t} = A_N \hat{v}_N + B_N f + \tilde{K}_N(y - \hat{y}), \tag{3.15b}$$

$$\hat{y} = C_N \hat{v}_N; \quad \hat{v}_N(0) = 0, \tag{3.15c}$$

where, due to the stabilizability and detectability of the ROM, we can adjust the controller gains $\tilde{G}_N$ and $\tilde{K}_N$ so that $A_N + B_N \tilde{G}_N$ and $A_N - \tilde{K}_N C_N$ have some stability margin. The controller (3.15) has finite dimension N (where $N = \dim H_N$) and consists of a linear feedback control law and full order state estimator (full order in the sense that it is matched to the full-order ROM). Much lower order controllers than (3.15) may be developed, but we will not pursue that here. Note that (3.15) has the form (2.12) where $\alpha = N$, $L_{11} = 0$, $L_{12} = \tilde{G}_N$, $L_{21} = \tilde{K}_N$, and $L_{22} = A_N + B_N \tilde{G}_N - \tilde{K}_N C_N$.

We define the *estimator error* $e_N \equiv \hat{v}_N - v_N$ and, from (3.14) and (3.15), obtain

$$\frac{\partial e_N}{\partial t} = (A_N - \tilde{K}_N C_N)\, e_N. \tag{3.16}$$

and

$$\frac{\partial v_N}{\partial t} = (A_N + B_N \tilde{G}_N)\, v_N + B_N \tilde{G}_N e_N. \tag{3.17}$$

If there were no solution error (i.e., $v = v_N$), then (3.16) and (3.17) would be designed with some stability margin. Consequently, the controller (3.15) would stabilize the model (3.14) *by design*; however, our principal concern in [5] was the closed-loop stability of the actual DPS (2.1) with the controller (3.15) when $v \neq v_N$, which is the usual case.

3.3. *Feedback Controllers: Indirect Model Reduction*

In the previous subsection, we have outlined the *direct approach* where a Galerkin approximation of the open-loop DPS (2.1) is made and a controller (3.15) based on this approximation is designed. The only requirement for doing this is that the ROM (A_N, B_N, C_N) in (3.14) be controllable and observable for each N. Now we present the *indirect approach* which is to Galerkin approximate the infinite-dimensional controller (2.10), (2.11).

We rewrite (2.10), (2.11) as the following:

$$f(t) = G\hat{v}(t), \tag{3.18a}$$

$$\frac{\partial \hat{v}(t)}{\partial t} = L\hat{v}(t) + Ky(t), \tag{3.18b}$$

$$\hat{v}(0) = 0, \tag{3.18c}$$

where $L = A + BG - KC$ is a closed operator with domain $D(L) = D(A)$ due to the fact that BG and KC are bounded (finite rank) perturbations of the closed operator A.

The Galerkin approximation of (3.18) is straightforward. We let

$$\hat{v}_N(t) = \sum_{k=1}^{N} \hat{v}_k(t)\,\psi_k, \tag{3.19}$$

where ψ_k are in H_N and consider

$$\begin{aligned} \frac{\partial \hat{v}_N}{\partial t} &= L\hat{v}_N + Ky + \hat{E}_N, \\ \hat{v}_N(0) &= 0. \end{aligned} \tag{3.20}$$

We choose the coefficients $\hat{v}_k$ so that

$$P_N(\hat{E}_N) = 0. \tag{3.21}$$

Although $\hat{v}_N \neq P_N\hat{v}$, we do have

$$\hat{v}_N = P_N\hat{v}_N. \tag{3.22}$$

From (3.20) and (3.22), we obtain the *Galerkin Feedback Controller*:

$$f(t) = G_N\hat{v}_N(t), \tag{3.23a}$$

$$\frac{\partial \hat{v}_N(t)}{\partial t} = L_N\hat{v}_N(t) + K_N\, y(t), \tag{3.23b}$$

$$\hat{v}_N(0) = 0, \tag{3.23c}$$

where $G_N \equiv GP_N$, $L_N \equiv P_NLP_N$, and $K_N \equiv P_NK$. This is also a finite-dimensional controller of the form (2.12) with $\alpha = N$, $L_{11} = 0$, $L_{12} = G_N$, $L_{21} = K_N$, and $L_{22} = L_N$. The difference between this controller (3.23) obtained via the indirect method and the one in (3.15) obtained by the direct method is the way the gains are obtained; the ones in (3.23) come from the infinite-dimensional stabilizing controller (3.18) [or (2.10), (2.11)], but the ones in (3.15) are calculated directly from the ROM (A_N, B_N, C_N) at each N.

3.4. *Equivalence of Direct and Indirect Approaches*

Consider the *closed-loop systems* (2.1) and (3.15),

$$\frac{\partial v}{\partial t} = Av + B\hat{G}_N \hat{v}_N, \tag{3.24a}$$

$$\frac{\partial v_N}{\partial t} = \tilde{K}_N Cv + \tilde{L}_N \hat{v}_N, \tag{3.24b}$$

where $\tilde{L}_N \equiv A_N + B_N \tilde{G}_N - \tilde{K}_N C_N$ and (2.1) and (2.23),

$$\frac{\partial \hat{v}}{\partial t} = Av + BG_N \hat{v}_N, \tag{3.25a}$$

$$\frac{\partial \hat{v}_N}{\partial t} = K_N Cv + L_N \hat{v}_N. \tag{3.25b}$$

We define the operators $\tilde{A}_N$ and $\bar{A}_N$ by

$$\tilde{A}_N = \begin{bmatrix} A & B\tilde{G}_N \\ \tilde{K}_N C & \tilde{L}_N \end{bmatrix} \tag{3.26}$$

and

$$\bar{A}_N = \begin{bmatrix} A & BG_N \\ K_N C & L_N \end{bmatrix}.$$

We shall say that the controllers (3.15) and (3.23) are *equivalent* if both exist and exponential stability of either (3.26) or (3.27) implies it for both. This is the case under the following conditions:

THEOREM 3.1. *Assume, for some N, that*

(a) *(A, B) and (A^*, C^*) have a pair of* orthogonal *stabilizing subspaces (H_N, H_R), i.e., $H_R = H_N^\perp$, and*

(b) *$\|A_{NR}\|$ and $\|A_{RN}\|$ are sufficiently small, where*

$$A_{RN} = P_R A P_N, \tag{3.28a}$$

$$A_{NR} = P_N A P_R. \tag{3.28b}$$

Then a stabilizing infinite-dimensional controller (3.18) *exists with gains G and K and if*

(c) *the gains $\tilde{G}_N$ and $\tilde{K}_N$ in* (3.15) *are chosen so that $\|\tilde{G}_N - G\|$ and $\|\tilde{K}_N - K\|$ are sufficiently small, then $A_N + B_N\tilde{G}_N$ and $A_N - \tilde{K}_N C_N$ are stable and, if either $\tilde{A}_N$ or $\bar{A}_N$ generates an exponentially stable C_0-semigroup, then both do.*

The proof of this result appears in Appendix I. In particular, if the stabilizing subspaces of (a) are also *reducing* (modal) subspaces, then Theorem 3.1 says that both controllers are equivalent when $\tilde{G}_N$ and $\tilde{K}_N$ are chosen close to GP_N and $P_N K$. This was also pointed out in [1].

Note that [2] indicates that (orthogonal) stabilizing subspaces must exist if a linear DPS is to be stablized by some finite-dimensional controller. Thus, hypothesis (a) is not at all strange. See also [11] which further substantiates this.

4.0. Convergence and Stability of the Indirect Model Reduction Method

In [5] we developed convergence and closed-loop stability results for the direct model reduction approach with Galerkin's method. With the indirect approach there will actually be a limiting stabilizing controller which is approached as our approximation improves; we will show this first and then indicate conditions under which some finite-dimensional approximating controller will yield a stable closed-loop system.

We will make the following two assumptions about the Galerkin method here:

$$P_N \xrightarrow{s} I \qquad \text{on} \quad D(A) \text{ or } H, \tag{4.1a}$$

$$A_N \equiv P_N A P_N \xrightarrow{s} A \qquad \text{on} \quad H, \tag{4.1b}$$

as $N \to \infty$. We have discussed (4.1a) already in Section 3.0; however, we note that, since $\|P_N\| = 1$ by orthogonality, we can assume (4.1a) is true on H even though defined only on $D(A)$ due to [8, Lemma 3.5, p. 151]. Assumption (4.1b) says that the Galerkin method *consistently approximates* the operator A. These assumptions are usually made whenever the Galerkin method is used to approximate partial differential equations. The following version of the *Trotter–Kato theorem* will be needed.

THEOREM 4.1. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of closed operators defined on $D(A)$ dense in H with "generalized" limit A (also closed), i.e.,*

$$R(\lambda, A_n) \xrightarrow{s} R(\lambda, A) \tag{4.2}$$

for some λ *such that* $\operatorname{Re}\lambda > \beta$, *and* A_n, A *generate* C_0*-semigroups* $U_n(t)$, $U(t)$, *respectively, satisfying for each* n,

$$\max(\|U_n(t)\|, \|U(t)\|) \leqslant Ke^{\beta t}, \qquad t \geqslant 0 \tag{4.3}$$

with $K \geqslant 1$ *and* β *real (both constants independent of* n*). Then*

$$U_n(t) \xrightarrow{s} U(t) \tag{4.4}$$

uniformly on any finite interval of $t \geqslant 0$.

The proof of this is given in [8, Theorem 2.16, p. 502]. Note that β need not be negative, but (4.3) does require a uniform exponential bound on $\|U_n(t)\|$, that is, independent of n.

Our main result gives conditions under which the indirect method (2.1) and (3.23) converges to the closed-loop system (2.1) and (3.18) which is stable:

THEOREM 4.2. *Assume* (4.1) *and* (A, B) *stabilizable and* (A, C) *detectable. If* $\bar{U}_N(t)$ *is the* C_0*-semigroup generated by* $\bar{A}_N$ *in* (3.27), *then it is uniformly exponentially bounded, i.e.*,

$$\|U_N(t)\| \leqslant Ke^{\beta t}, \qquad t \geqslant 0, \tag{4.5}$$

where $K \geqslant 1$ *and* β *real (both independent of* N*) and* $\bar{U}_N(t) \to^s \bar{U}(t)$ *uniformly on any finite interval of* $t \geqslant 0$, *where* $\bar{U}(t)$ *is the* C_0*-semigroup generated by*

$$\bar{A} = \begin{bmatrix} A & BG \\ KC & L \end{bmatrix}, \tag{4.6}$$

where $A + BG$ *and* $A - KC$ *are (exponentially) stable.*

The proof uses Theorem 4.1 and is given in Appendix II. It depends on noting

$$\bar{A}_N = \bar{P}_N \bar{A} \bar{P}_N, \tag{4.7}$$

where

$$\bar{P}_N \equiv \begin{bmatrix} I & 0 \\ 0 & P_N \end{bmatrix} \tag{4.8}$$

is an orthogonal projection on the Hilbert space $\bar{H} = H \times H$ with corresponding complementary projection

$$\bar{P}_R \equiv \begin{bmatrix} 0 & 0 \\ 0 & P_R \end{bmatrix}. \tag{4.9}$$

From (4.1), we have as $N \to \infty$

$$\bar{P}_N \xrightarrow{s} I, \tag{4.10a}$$

$$\bar{P}_R \xrightarrow{s} 0. \tag{4.10b}$$

Also, it is clear that $\bar{A}$ generates $\bar{U}(t)$ exponentially stable because

$$\bar{Q}^{-1}\bar{A}\bar{Q} = \begin{bmatrix} A+BG & BG \\ 0 & A-KC \end{bmatrix} \tag{4.11}$$

is (exp.) stable by the choice of the infinite-dimensional controller gains G and K where

$$\bar{Q} \equiv \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \quad \text{and} \quad \bar{Q}^{-1} \equiv \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}. \tag{4.12}$$

Thus,

$$\|U(t)\| \leqslant Ke^{-\sigma t}, \qquad t \geqslant 0, \tag{4.13}$$

where $\sigma > 0$ and we use the same K as in (4.5) without loss of generality.

Note that (4.5) does not require uniform exponential *stability* for $\bar{A}_N$. Yet we are most interested in the question: When does $\bar{A}_N$ become exponentially stable for sufficiently large N? The results of Theorem 4.2 cannot answer this question because they are only valid on *finite* intervals of $t \geqslant 0$. Let $\tilde{e}_N \equiv \hat{v}_N - P_N\hat{v}$, where $\hat{v}_N$ and $\hat{v}$ are given in (3.23) and (3.18), respectively; from (3.23) and (3.18) we have

$$\begin{aligned} \frac{\partial e_N}{\partial t} &= L_N\hat{v}_N + K_N y - P_N[L\hat{v} + Ky] \\ &= L_N\tilde{e}_N - P_N L P_R \hat{v} + [K_N - P_N K]\, y \\ &= L_N\tilde{e}_N - L_{NR}\hat{v} \end{aligned} \tag{4.14}$$

because $K_N \equiv P_N K$ and $L_{NR} \equiv P_N L P_R$.

The *closed-loop system* (3.25) becomes (using (4.14) and (3.18)):

$$\frac{\partial v}{\partial t} = Av + BG_N\hat{v} + BG_N\tilde{e}_N, \tag{4.15a}$$

$$\frac{\partial \hat{v}}{\partial t} = KCv + L\hat{v}, \tag{4.15b}$$

$$\frac{\partial \tilde{e}_N}{\partial t} = -L_{NR}\hat{v} + L_N\tilde{e}_N. \tag{4.15c}$$

Equivalently, with

$$\omega \equiv \begin{bmatrix} v \\ \hat{v} \\ \tilde{e}_N \end{bmatrix},$$

we have

$$\frac{\partial \omega}{\partial t} = \tilde{A}_N \omega, \tag{4.16}$$

where

$$\tilde{A}_N \equiv \begin{bmatrix} A & BG_N & BG_N \\ KC & L & 0 \\ 0 & -L_{NR} & L_N \end{bmatrix}.$$

The following result gives conditions under which the closed-loop system consisting of (2.1) and (3.23) will be stable for sufficiently large N:

THEOREM 4.3. *Assume* (4.1a) *and* (A, B) *stabilizable and* (A, C) *detectable* (*with* $A + BG$ *and* $A - KC$ *stable*). *If*

(a) $\|A_{NR} v\| \leqslant \alpha_N \|v\|$ *for all* v *in* $D(A)$ *with* $\lim_N \alpha_N = 0$, (4.17)

(b) L_N uniformly *exponentially stable for N sufficiently large, i.e.,*

$$\|V_N(t)\| \leqslant K_0 e^{-\sigma_0 t}, \qquad t \geqslant 0, \tag{4.18}$$

where $V_N(t)$ *is the* C_0*-semigroup generated by* L_N *and* $K_0 \geqslant 1$, $\sigma_0 > 0$ (*independent of* N), *then, for* N *sufficiently large, the* C_0*-semigroup* $\tilde{U}_N(t)$ *generated by* $\tilde{A}_N$ *in* (4.16) *is uniformly exponentially stable, i.e.,*

$$\|\tilde{U}_N(t)\| \leqslant \tilde{K} e^{-\tilde{\sigma}_N t}, \qquad t \geqslant 0, \tag{4.19}$$

where $\tilde{K} \geqslant 1$ (*independent of* N), $\tilde{\sigma}_N \leqslant \tilde{\sigma}$, *and* $\lim_N \tilde{\sigma}_N = \tilde{\sigma} \equiv \min(\sigma, \sigma_0)$ *with* σ and σ_0 *given in* (4.13) *and* (4.18), *respectively. Furthermore, the results of Theorem* 4.2 *hold.*

The proof is given in Appendix III. Note that $A_{NR} \equiv P_N A P_R$ is a bounded operator on $D(A)$ for each N. However, (4.17) says that those bounds converge to zero; this is a *uniform consistency* requirement for the Galerkin method on the differential operator A in (2.1). Also, note that the above result says that $\hat{v}_N \to_N \hat{v}$ for the controller (3.23).

If *reducing* (modal) *subspaces* for A are used for the Galerkin method, then $A_{NR} = 0$; hence, (a) is satisfied. Thus, we have closed-loop stability for

N large when (b) is satisfied. However, note that for reducing subspaces: $AP_N = P_N A$; also, P_N commutes with the C_0-semigroup generated by A. Suppose (H_N, H_R) reducing subspaces are also stabilizing subspaces for (A, B) and (A^*, C^*), then for (b) to hold we need only that $L = A + BG - KG$ is exponentially stable, i.e., we have a *stable*, infinite-dimensional controller, and $BG - KG$ is invariant on H_N.

One may object to hypothesis (b), but to avoid it we must appeal to estimates like the following:

$$\bar{A}_N = \bar{A} + \bar{\Delta}_N, \tag{4.20}$$

where

$$\bar{\Delta}_N \equiv \begin{bmatrix} 0 & -BG_R \\ -K_R C & L_N - L \end{bmatrix}$$

with $\bar{A}$ exponentially stable and $\bar{\Delta}_B \to^s 0$ due to either (4.16) or (4.1a) and (4.17) because $A_N - A = P_R A + A_{NR} \to^s 0$ implies $L_N - L \to^s 0$ but *not necessarily uniformly*. Therefore, $\bar{\Delta}_N$ is uniformly bounded in N:

$$\|\bar{\Delta}_N\| \leqslant \Gamma < \infty \tag{4.21}$$

but, we can only obtain closed-loop stability via [10, Theorem 10.9] when

$$\Gamma < \sigma/K,$$

where σ, K come from (4.13). Better estimates to relax the hypotheses (a) and (b) are desirable; we hope they will be forthcoming.

APPENDIX I: Proof of Theorem 3.1

Since (H_N, H_R) are stabilizing subspaces for (A, B) and (A^*, C^*), we have G and K such that $A + BG$ and $A - KC$ exp. stable with $GP_R = 0$ and $P_R K = 0$ (because, since $P_R^* = P_R$, $0 = K^* P_R = (P_R K)^*$ and $P_R K = \overline{P_R K} = (P_R K)^{**} = O^* = 0$). This guarantees existence of (3.18) by Theorems 2.1 and 2.2 and we can apply the indirect method. Furthermore, we have

$$A + BG = \begin{bmatrix} A_N + B_N G_N & A_{NR} \\ A_{RN} + B_R G_N & A_R \end{bmatrix}$$

$$A - KC = \begin{bmatrix} A_N - K_N C_N & A_{NR} - K_N C_R \\ A_{RN} & A_R \end{bmatrix},$$

where $G_N = GP_N$ and $K_N = P_N K$ generate exp. stable C_0-semigroups. Consequently, using [10, Theorem 10.9], we have $A_N + B_N G_N =$

$P_N(A+BG)P_N$, $A_N - K_N C_N = P_N(A-KC)P_N$, and $A_R = P_R A P_R$ exp. stable due to (b). Thus, if (c) is satisfied, then $A_N + B_N \tilde{G}_N$ and $A_N - \tilde{K}_N C_N$ are stable; hence (A_N, B_N, C_N) stabilizable and detectable and we can apply the direct method.

From $GP_R = 0$ and $P_R K = 0$, we see that $G = GP_N = G_N$ and $K = P_N K = K_N$. Consequently, from (3.27)

$$\bar{A}_N = \begin{bmatrix} A & BG \\ KC & L_N \end{bmatrix},$$

where $L_N A_N + B_N G - KC_N$. Now

$$\bar{A}_N - \tilde{A}_N = \left[\begin{array}{c|c} 0 & B(G-\tilde{G}_N) \\ \hline (K-\tilde{K}_N)\,C & P_N[B(G-\tilde{G}_N) - (K-\tilde{K}_N)\,C]\,P_N \end{array}\right]$$

and this is a bounded operator ($\|P_N\| = 1$ since it is orthogonal) which is small when (c) is satisfied. Therefore, using [10, Theorem 10.9] again, we have the desired result.

APPENDIX II: Proof of Theorem 4.2

Since (4.1a) holds, we have $G_N \equiv GP_N \to^s G$ and $K_N \equiv P_N K \to^s K$ on H. Furthermore, from (4.1b) and the above,

$$L_N \equiv P_N L P_N = A_N + B_N G_N - K_N C_N \xrightarrow{s} L \qquad \text{on } H.$$

Therefore, from (4.7), $\bar{A}_N \to^s \bar{A}$ on $D(A)$.

Consider, from (3.27) and (4.6),

$$\bar{A}_N = \bar{A} + \bar{\Delta}_N, \tag{A.II.1}$$

where

$$\bar{\Delta}_N \equiv \begin{bmatrix} 0 & B(G_N - G) \\ (K_N - K)\,C & L_N - L \end{bmatrix}.$$

From the above argument, $\bar{\Delta}_N \to^s 0$ on $\bar{H}$. Thus by [8, pp. 150; 151] (i.e., use of the uniform Boundedness Principle), $\bar{\Delta}_N$ is uniformly bounded:

$$\|\bar{\Delta}_N\| \leqslant \Gamma < \infty \qquad \text{for all } N, \tag{A.II.2}$$

where Γ is indep. of N. Hence, by (4.13) and [10] Theorem 10.9, we have (4.5) with $\beta \equiv -\sigma + K\Gamma$ real. Now consider for λ in $\rho(\bar{A}_N) \cap \rho(\bar{A})$:

$$R(\lambda, \bar{A}_N) - R(\lambda, \bar{A}) = R(\lambda, \bar{A}_N)[\bar{A}_N - \bar{A}]\,R(\lambda, \bar{A}) \tag{A.II.1}$$

which is obtained by multiplying the above on the left by $(\lambda I - \bar{A}_N)$ and right by $(\lambda I - A)$.

From the previous discussion, $\beta > -\sigma$ in (4.5) and (4.13) and we choose λ real with $\lambda > \beta$, then λ is in $\rho(\bar{A}_N) \cap \rho(\bar{A})$. From (4.5) and the Hille–Yosida theorem (see (2.6), (2.7)),

$$\|R(\lambda, \bar{A}_N)\| \leqslant \frac{K}{\lambda - \beta},$$

therefore,

$$\|[R(\lambda, \bar{A}_N) - R(\lambda, \bar{A})]\, v\| \leqslant \frac{K}{\lambda - \beta} \|(\bar{A}_N - \bar{A})\, \omega\|,$$

where v is in $\bar{H}$ and $\omega \equiv R(\lambda, \bar{A})\, v$. Because ω is in $D(\bar{A})$ and $\bar{A}_N \to^s \bar{A}$ there, this gives us $R(\lambda, \bar{A}_N) \to^s R(\lambda, \bar{A})$. Also, since $\beta > -\sigma$, $\|\bar{U}(t)\| \leqslant Ke^{-\sigma t} \leqslant Ke^{\beta t}$. Thus, the hypotheses of Theorem 4.1 are satisfied and the desired result follows.

APPENDIX III: Proof of Theorem 4.3

Since $D(A)$ is dense in H, we have from (4.17) that

$$\|A_{NR}\| \leqslant \alpha_N \xrightarrow[N]{} 0. \tag{A.III.1}$$

Also $A - A_N = A - P_N A P_N + A_{NR} - A_{NR} = A - P_N A + A_{NR} = P_R A + A_{NR}$, we have $A_N \to^s A$ on H because of (4.1a) and (4.14); thus (4.16) is satisfied and Theorem 4.2 holds.

From (4.16), we have

$$\tilde{A}_N = \tilde{\tilde{A}}_N + \tilde{\Delta}_N,$$

where

$$\tilde{\tilde{A}}_N \equiv \begin{bmatrix} A & BG & BG \\ KC & L & 0 \\ 0 & 0 & L_N \end{bmatrix}$$

and

$$\tilde{\Delta}_N \equiv \begin{bmatrix} 0 & -BG_R & -BG_R \\ 0 & 0 & 0 \\ 0 & -L_{NR} & 0 \end{bmatrix}$$

with $G_R \equiv GP_R$. Since $\bar{A} = \left[\begin{smallmatrix} A & BG \\ KC & L \end{smallmatrix}\right]$ satisfies (4.13) and, for N large, L_N satisfies (4.18), we have the C_0-semigroup $\tilde{\tilde{U}}_N(t)$ generated by $\tilde{\tilde{A}}_N$ satisfying

$$\|\tilde{\tilde{U}}_N(t)\| \leqslant \tilde{K} e^{-\tilde{\sigma} t}, \qquad t \geqslant 0.$$

Also, $BG_R \to^s 0$ due to (4.1a), but again, since this is a finite rank operator, we have $\lim_N \|BG_R\| = 0$. In addition from similar arguments

$$\lim_N \|B_N G_R\| = 0,$$

$$\lim_N \|K_N C_R\| = 0,$$

therefore

$$\lim_N \|L_{NR}\| = \lim_N \alpha_N = 0$$

due to $L_{NR} = P_N L P_R = A_{NR} + B_N G_R - K_N C_R$ and (4.17). Consequently, we have that $\lim_N \|\tilde{\Delta}_N\| = 0$. So, by [10, Theorem 10.9], we have $\tilde{U}_N(t)$ satisfying (4.19) with $\tilde{\sigma}_N \equiv \tilde{\sigma} - \tilde{K} \|\tilde{\Delta}_N\| \to_N \tilde{\sigma}$ which is the desired result.

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